In conclusion, we note that the functions (2.1) can be written in the form of a differential operator [3], provided that we put $W = F'(\zeta)$ where $F(\zeta)$ is an arbitrary analytic function of the argument ζ (2.2).

REFERENCES

1. GOL'DENVEIZER A. L., *Theory of Thin Elastic Shells*. Nauka, Moscow, 1976.
2. VLASOV V. Z., *The General Theory of Shells and its Applications in Technology*. Gostekhizdat, Moscow, 1949.
3. NAZAROV G. I., The exact solution of the equations of gas dynamics. *Izv. Akad. Nauk. SSSR, MZhG* 3, 1968.

Translated by L.K.

J. Appl. Maths Mechs Vol. 55, No. 5, pp. 746–750, 1991
Printed in Great Britain.
0021–8928/91 $15.00+.00

INTERACTION OF DISLOCATIONS IN AN ANISOTROPIC MEDIUM†

S. V. KUZNETSOV

Moscow

(*Received* 28 *June* 1989)

THE INTERACTION of two dislocations at a distance from each other, in an anisotropic medium with elastic anisotropy of general type is considered and the forces acting on the dislocation defects determined. The solution is constructed using the method of multipolar expansions.

The mechanical properties of crystalline bodies depend to considerable degree on the presence of defects within them and their interaction with each other. The energy of interaction between the defects is the basic factor determining their mutual distribution and orientation within the crystal.

The publications available in this field deal either with the determination of the energy at which an isolated dislocation loop appears [1–3], with the associated forces acting on an isolated defect [4] and with the stress fields near the dislocation loop [5], or with the study of the interaction of dislocation defects between each other [6–10] or with foreign inclusions [11–13].

In spite of the fact that the majority of crystals are elastically anisotropic, the investigations leading to the determination of the energy of interaction between the dislocations were carried out, basically, for isotropic of transversally isotropic (hexagonal) crystals. This can be explained by the need to use the fundamental solutions of the equations of equilibrium, whose derivatives are used, in the majority of cases, to express the energy of interaction between the dislocations.

The fundamental solutions of the equations of equilibrium are constructed in closed form for isotropic media (Kelvin solution) and for a subclass of orthotropic materials, which includes transversally isotropic media, in

† *Prikl. Mat. Mekh.* Vol. 55, No. 5, pp. 870–873, 1991.

[14]. In the general case of elastic anisotropy only numerical methods of constructing the fundamental solutions are known [15].

The approach employed in [15] is used below to obtain an expression for determining the energy and the forces of interaction of two dislocation loops separated from each other in an anisotropic medium with a general type of anisotropy.

1. FUNDAMENTAL EQUATIONS, OPERATORS AND SYMBOLS

We consider an elastic anisotropic medium whose equations of statics (equilibrium) have the form

$$\mathbf{A}\,(\partial_x)\mathbf{u} \equiv -\operatorname{div}\,\mathbf{C}\cdot\cdot\,(\nabla\mathbf{u}) \tag{1.1}$$

where $\mathbf{u}$ is the displacement vector and $\mathbf{C}$ is a tetravalent, strictly elliptical elasticity tensor:

$$(\boldsymbol{\eta}\otimes\boldsymbol{\xi})\cdot\cdot\mathbf{C}\cdot\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\eta}) > 0,\quad \forall\boldsymbol{\eta},\ \boldsymbol{\xi}\in R^3,\ \boldsymbol{\eta},\ \boldsymbol{\xi}\neq 0 \tag{1.2}$$

We assume that the medium in question is hyperelastic, which ensures the symmetry of the tensor $\mathbf{C}$ with respect to the terminal pair of indices: $C^{mnij} = C^{ijmn}$. Condition (1.2) ensures the ellipticity of the matrix symbol $\mathbf{A}^*$:

$$\mathbf{A}^*\,(\boldsymbol{\xi}) = (2\pi)^2\boldsymbol{\xi}\cdot\mathbf{C}\cdot\boldsymbol{\xi} \tag{1.3}$$

obtained by applying the Fourier transformation

$$f^*\,(\boldsymbol{\xi}) = \int\limits_{R^3} f\,(\mathbf{x})\,\exp\,(-2\pi i\,\mathbf{x}\cdot\boldsymbol{\xi})dx,\ f\in L^2\,(R^3)$$

to Eq. (1.1).

Using the symbol $\mathbf{A}^*$ we can easily construct the inverse control $\mathbf{E}^*$ which represents the following Fourier-transformed basic solution:

$$\mathbf{E}^*\,(\boldsymbol{\xi}) = \mathbf{A}_0{}^*\,(\boldsymbol{\xi})/\det\,\mathbf{A}^*\,(\boldsymbol{\xi}) \tag{1.4}$$

where $\mathbf{A}_0{}^*$ is the matrix of the cofactors of the symbol $\mathbf{A}^*$. Formula (1.4) shows that the symbol $\mathbf{E}^*$ is elliptical, real-analytic everywhere in $R^3\backslash 0$ and homogeneous in $|\xi|$, of degree -2. Henceforth we shall also require the stress operator symbol

$$\mathbf{T}_\nu{}^*\,(\boldsymbol{\xi}) = 2\pi i\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\xi} \tag{1.5}$$

where $\boldsymbol{\nu}$ is the vector of the unit normal to the surface under investigation. From (1.5) it follows that, in the case of a hyperelastic material, the transposed symbol $\mathbf{T}_\nu{}^{*t}$ has the form

$$\mathbf{T}_\nu^{*t}\,(\boldsymbol{\xi}) = 2\pi i\boldsymbol{\xi}\cdot\mathbf{C}\cdot\boldsymbol{\nu} \tag{1.6}$$

2 REPRESENTATION OF THE SOLUTION

The displacement field in a medium containing an isolated dislocation, can be written in the form of a double layer potential

$$\mathbf{u}\,(\mathbf{x}) = \int\limits_{\Omega_1} \mathbf{b}_1\cdot\mathbf{T}_{\nu_1}(\partial_y)\,\mathbf{E}\,(\mathbf{x}-\mathbf{y}')\,dy', \tag{2.1}$$

where $\mathbf{b}_1$ is the Burgers vector given in Ω_1 and Ω_2 is a plane region bounded by the contour $\partial\Omega_1$ representing the dislocation loop.

Note. The representation (2.1) is actually identical with the expression obtained by Volterra [16] for describing the displacements in a crystal containing a given dislocation. In [16] the general case was studied when the vector $\mathbf{b}_1$ could be variable and the dislocation zone represented a part of some curvilinear surface in R^3. Following this, Burgers in [17] proposed that dislocations with constant vector $\mathbf{b}_1$, called the Burgers vector, should be discussed. For a constant vector $\mathbf{b}_1$ the representation (2.1) is invariant with respect to the surface carrying the dislocation. It is for this reason that we can speak, in the case when $\partial\Omega_1$ is a plane closed curve, of a plane region carrying the dislocation. In [17] and later in [18], formulas were obtained for an isotropic elastic medium, reducing the integral over Ω_1 in (2.1) to a contour integral over $\partial\Omega_1$. No such formulas are known for the case of an anisotropic medium with a general type of anisotropy.

The stress in the plane in which the second dislocation loop Ω_2 is situated, is given by the formula

$$\mathbf{t}(\mathbf{x}) = \mathbf{T}_{\nu_2}(\partial_x) \int_{\Omega_1} \mathbf{b}_1 \cdot \mathbf{T}_{\nu_1}(\partial_y)\, \mathbf{E}(\mathbf{x}-\mathbf{y}')\, dy', \qquad \mathbf{x} \in \Omega_2 \tag{2.2}$$

The energy of interaction between two distant dislocation loops is the work necessary to form the second dislocation loop within the stress field generated by the first loop [9, 10], i.e.

$$W = -\int_{\Omega_2} \mathbf{b}_2 \cdot \mathbf{t}(\mathbf{x}')\, dx' \approx -\,\mathrm{mes}\,(\Omega_1)\,\mathrm{mes}\,(\Omega_2)\, \mathbf{b}_2 \cdot \mathbf{T}_{\nu_2}(\partial_x)\, \mathbf{b}_1 \cdot \mathbf{T}_{\nu_1}(\partial_y)\, \mathbf{E}(\mathbf{x}_2 - \mathbf{y}_1) \tag{2.3}$$

where $\mathbf{x}_2$, $\mathbf{y}_1$ are the coordinates of the "centres" of dislocations Ω_1 and Ω_2 respectively. Henceforth it will be convenient to place $\mathbf{y}_1$ at the origin of coordinates in R^3.

Expression (2.3) contains the unknown fundamental solution $\mathbf{E}$. In order to obtain the necessary computational formulas we shall consider the symbol of the integrodifferential operator

$$\mathbf{G}_{\nu_1,\nu_2}(\mathbf{x}-\mathbf{y}) = -\mathbf{T}_{\nu_2}^{\,t}(\partial_x)\mathbf{T}_{\nu_1}(\partial_y)\, \mathbf{E}(\mathbf{x}-\mathbf{y}) \tag{2.4}$$

When (1.5) and (1.6) are taken into account, the symbol $\mathbf{G}^*_{\nu_1,\nu_2}$ takes the form

$$\mathbf{G}^*_{\nu_1,\nu_2}(\boldsymbol{\xi}) = 4\pi^2\, (\boldsymbol{\nu}_2 \cdot \mathbf{C} \cdot \boldsymbol{\xi}) \cdot \mathbf{E} \cdot (\boldsymbol{\xi} \cdot \mathbf{C} \cdot \boldsymbol{\nu}_1) \tag{2.5}$$

3. THE ENERGY AND FORCES OF INTERACTION

When the material of the medium in question is hyperelastic and $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_2$, i.e. when the dislocations have the same orientation, the following assertion holds:

Assertion 1. The symbol $\mathbf{G}^*_{\nu_1,\nu_2}$ is positive semi-definite.

Proof. The proof is based on the strict ellipticity of the symbol $\mathbf{E}^*$ ensuring that the following inequality holds:

$$\boldsymbol{\eta} \cdot \mathbf{G}^*_{\nu_1,\nu_2} \cdot \boldsymbol{\eta} = 4\pi^2\, ((\boldsymbol{\eta} \otimes \boldsymbol{\nu}) \cdot\cdot\, \mathbf{C} \cdot \boldsymbol{\xi}) \cdot \mathbf{E}^*(\boldsymbol{\xi}) \cdot (\boldsymbol{\xi} \cdot \mathbf{C} \cdot (\boldsymbol{\nu} \otimes \boldsymbol{\eta})) \geqslant 0$$

Assertion 2. The operator $\mathbf{G}^*_{\nu_1,\nu_2}$ can be represented in the form $\mathbf{G}_{\nu_1,\nu_2} = \mathbf{G}^\circ_{\nu_1,\nu_2} + \mathbf{G}^1_{\nu_1,\nu_2}$, where $\mathbf{G}^\circ_{\nu_1,\nu_2}$ is a constant operator (matrix) and $\mathbf{G}^1_{\nu_1,\nu_2}$ is the singular matrix operator.

Proof. Since the symbol $\mathbf{E}^*$ of degree -2 is homogeneous in $|\xi|$, it follows that the symbol $\mathbf{G}^*_{\nu_1,\nu_2}$ is homogeneous of degree 0. Assuming that $\mathbf{G}^*_{\nu_1,\nu_2}$ is real-analytic in $R^3\backslash 0$ and that Marcinkiewicz theorem on multiplicators holds, we obtain the result required. Also, if the mean value (on integrating over the unit sphere in R^3) of any one component of the symbol $\mathbf{G}^*_{\nu_1,\nu_2}$ is non-zero, then the corresponding component of $\mathbf{G}^\circ_{\nu_1,\nu_2}$ will also be non-zero.

Taking (2.3) into account, we can write the formula for the energy in the form

$$W \approx \mathrm{mes}\,(\Omega_1)\mathrm{mes}\,(\Omega_2)\; \mathbf{b}_2 \cdot \mathbf{G}^1_{\nu_1,\nu_2} \cdot \mathbf{b}_1 \tag{3.1}$$

The specific form of the singular kernel $\mathbf{G}^1_{\nu_1,\nu_2}$ is found as follows.

Let the following expansion of the symbol $\mathbf{G}^*_{\nu_1,\nu_2}$ in series in spherical harmonics be given:

$$\mathbf{G}^*_{\nu_1,\nu_2}(\xi') = \sum_{p=0,2\ldots}^{\infty} \sum_{k=1}^{2p+1} \mathbf{G}^{p,k}_{\nu_1,\nu_2} Y_k{}^p(\xi'), \quad \xi' \in S \tag{3.2}$$

where $Y_k{}^p$ are spherical harmonics and the matrix coefficients $\mathbf{G}^{p,k}_{\nu_1,\nu_2}$ are determined by integrating over a sphere of unit radius in R^3

$$\mathbf{G}^{p,k}_{\nu_1,\nu_2} = (2\pi)^{-2} \int_S \mathbf{G}^*_{\nu_1,\nu_2}(\xi') Y_k{}^p(\xi')\, d\xi'$$

The fact that only harmonics of even degree appear in the expansion (3.2) is due to the positive homogeneity of the symbol $\mathbf{G}^*_{\nu_1,\nu_2}$. An inverse Fourier transformation of (3.2) yields

$$\mathbf{G}^1_{\nu_1,\nu_2}(\mathbf{x}) = \pi^{-3/2} \sum_{p=2,4,\ldots}^{\infty} (-1)^{p/2} \frac{\Gamma((p+3)/2)}{\Gamma(p/2)} \sum_{k=1}^{2p+1} \mathbf{G}^{p,k}_{\nu_1,\nu_2} \frac{Y_k{}^p(\mathbf{x}')}{|x|^3} \tag{3.3}$$

The force of interaction between the dislocations $F = \nabla W$ is related to the energy of interaction. Inspection of the radial component $F_r = -(\nabla W \cdot \mathbf{r}) = -\partial_r W$ shows that when $F_r < 0$, the dislocations attract each other and repel each other when $F_r > 0$. Taking (3.1) and representation (3.3) into account, we see that the component F_r is directly proportional to the dislocation energy:

$$F_r = -3W/|x| \tag{3.4}$$

and the asymptotic estimate $F_r = O(|x|^{-4})$, $|x| \to \infty$ holds for F_r. Thus, in order to determine the zones of mutual attraction or repulsion we must, in accordance with (3.4), calculate the interaction energy on a unit sphere S in R^3.

Expression (3.4) yields the following expression for the tangential component:

$$F_\theta = \mathbf{F} - F_r\mathbf{x} = -\nabla W - 3W|x|^{-2}\mathbf{x} \tag{3.5}$$

with asymptotic estimate $F_\theta = O(|x|^{-3})$, $|x| \to \infty$.

If we assume that W is unimodal within the limits of the constant sign zones on S, then the formula $\mathbf{F} = -\nabla W$ will determine the motion of the dislocation loop Ω_2 relative to Ω_1. For the loop Ω_2 situated within the zone of mutual attraction ($W > 0$) we shall have its motion towards Ω_1 with simultaneous translation into the zone of the nearest (negative) relative minimum of W. After this we shall have, in accordance with (3.4), motion along the radius from the loop Ω_1.

Using the above analysis and the asymptotic estimates following from formulas (3.4) and (3.5), we can conclude that dislocation walls must form on the lines characterized by the directions on which W reaches relative minima. This result, which generalizes the corresponding investigations in [8, 10], holds for media with arbitrary elastic anisotropy.

REFERENCES

1. NABARRO F. R. N., The mathematical theory of stationary dislocations. *Adv. Phys.* **1**, 3, 271–394, 1952.
2. CHOU Y. T. and ESHELBY J. D., The energy and line tension of a dislocation in a hexagonal crystal. *J. Mech. Phys. Solids* **10**, 1, 27–34, 1962.
3. KRÖNER E., *General Continuum Theory of Dislocations and Eigen Stresses*. Mir, Moscow, 1965.
4. BATRA R. C., The force on a lattice defect in an elastic body. *J. Elast.* **17**, 1, 3–8, 1987.
5. OHR S. M., Anisotropic elasticity of a prismatic dislocation loop in a hexagonal crystal. *J. Appl. Phys.* **43**, 4, 1361–1365, 1972.
6. COTTRELL A. H., *Dislocation and Plastic Flow Crystals*. Clarendon Press, Oxford, 1956.
7. CHOU Y. T., Interaction of parallel dislocations in a hexagonal crystal. *J. Appl. Phys.* **33**, 9, 2747–2751, 1962.
8. CHOU Y. T., The energy of circular dislocation loops in thin plates. *Acta Met.* **11**, 8, 829–834, 1963.
9. WILLIS J. R., The elastic interaction energy of dislocation loops in anisotropic media. *Quart. J. Mech. Appl. Math.* **18**, 4, 419–433, 1965.

10. TUPHOLME C. E., Dislocation loops in hexagonal crystals. *J. Mech. Phys. Solids* **22,** 4, 309–321, 1974.
11. DUNDURS J. and MURA T., Interaction between an edge dislocation and two circular inclusions. *J. Mech. Phys. Solids* **12,** 3, 177–189, 1964.
12. FUKUZAKI K. and SHIOYA S., On the interaction between an edge dislocation and two circular inclusions in an infinite medium. *Int. J. Engng Sci.* **24,** 12, 1771–1787, 1986.
13. FUKUZAKI K. and SHIOYA S., On the interaction between an edge dislocation and two circular inclusions in an infinite medium (continued report). *Int. J. Engng Sci.* **25,** 8, 1017–1027, 1987.
14. KRÖNER E., Das Fundamentalintegral der anisotropen elastichen Differentialgleichungen. *Z. Phys* **136,** 4, 402–410, 1953.
15. KUZNETSOV S. V., Fundamental solutions of the Lamé equations for anisotropic media. *Izv. Akad. Nauk SSSR*, MTT 4, 1989.
16. VOLTERRA V., Sur l'equilibre des corps multiplielement connexes. *Ann. Ecol. Norm. Super.* **24,** 401–517, 1907.
17. BURGERS J. M., Some considerations of the field of stress connected with dislocations in a regular crystal lattice. *Proc. Kon. Neder. Akad. Wetensch.* **42,** 293–325; 378–399, 1939.
18. PEACH M. O. and KOEHLER J. S., The forces exerted in dislocations and the stress field produced by them. *Phys. Rev. Ser. 2* **80,** 3, 436–439, 1950.

Translated by L.K.

J. Appl. Maths Mechs Vol. 55, No. 5, pp. 750–752, 1991
Printed in Great Britain.

0021-8928/91 $15.00+.00

RAYLEIGH MOTIONS IN AN ELASTIC HALF-SPACE WITH A CONSTRAINED BOUNDARY†

V. S. LENSKII

Moscow

(*Received* 25 *June* 1990)

STATIONARY surface waves in elastic half-space with boundary conditions corresponding to a combination of the Winkler model and an inertial layer at the boundary are studied. It is found that the velocity of propagation of a harmonic wave depends on the frequency, and the presence of constraints in a direction normal to the boundary results in stopping of the low frequencies, when the effect of elastic rigidity and inertia of the boundary are taken into account at the same time, and when the inertia of the support has no effect. The frequencies are not stopped when the displacements along the boundary are restricted and when the influence of elastic rigidity on the normal displacements of the boundary is neglected.

† *Prikl. Mat. Mekh.* Vol. 55, No. 5, pp. 873–875, 1991.